# Uniform exponential growth for some SL(2,R) matrix products

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# UNIFORM EXPONENTIAL GROWTH FOR SOME $SL(2,\mathbb{R})$ MATRIX PRODUCTS

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ABSTRACT. Given a hyperbolic matrix $H \in SL(2,\mathbb{R})$, we prove that for almost every $R \in SL(2,\mathbb{R})$, any product of length $n$ of $H$ and $R$ grows exponentially fast with $n$ provided the matrix $R$ occurs less than $o(\frac{n}{\log n \log\log n})$ times.

## 1. Introduction

For $t, \theta \in \mathbb{R}$, let $H = H(t)$ be the hyperbolic matrix $\begin{pmatrix} \exp \frac{1}{2}t & 0 \\ 0 & \exp -\frac{1}{2}t \end{pmatrix}$ and let $R = R(\theta)$ be the rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. For a finite word $w = w_n \dots w_1$ on the symbols $H$ and $R$, we let $|w|$ denote its length and we let $m(w)$ denote the number of occurrences of $R$ in $w$. For any such word, and for any choice of parameters $t$ and $\theta$, we let $A_w(t,\theta)$ denote the corresponding matrix product in $\mathrm{SL}(2,\mathbb{R})$, and denote by $\|A_w(t,\theta)\|$ its norm.

By the Oseledets Theorem, for a typical large word $w$ on $H$ and $R$, the size of the matrix product is given up to subexponential error, by $e^{L(t,\theta)|w|}$, where $L(t,\theta)$ is the Lyapunov exponent of the Bernoulli product giving equal probabilities for $H$ and $R$. By Furstenberg's Theorem (cf [3]), $L(t,\theta) > 0$ unless $t = 0$ or $\theta = \pi/2 \mod \pi$, thus hyperbolic behavior prevails under a very mild "transversality condition" on the pair $(H,R)$.

Here we are interested in the following subtler question: Assuming some stronger transversality condition on the pair $(H,R)$, can one ensure hyperbolic behavior just by limiting the frequency of rotation elements in the word? A basic question in this direction, raised by Bochi and Fayad in [1], is whether for almost every $t$ and $\theta$, a condition of the type $C(t,\theta)m(w) \leq |w|$ implies that $\|A_w(t,\theta)\|$ grows exponentially. While this question is still open, in [2], Fayad and Krikorian showed that for almost every $t$ and $\theta$, one has exponential growth provided $m(w) \leq |w|^\alpha$ with $0 < \alpha < 1/2$. Our goal in this paper will be to show that the weaker condition $C(t,\theta)m(w)\log m(w) \log\log m(w) \leq |w|$ suffices.

**Theorem 1.** *For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H$ and $R$, if $m(w) \leq \epsilon|w|(\log|w| \log\log|w|)^{-1}$, then the spectral radius of $A_w(t,\theta)$ is at least $e^{|w|\gamma}$.*

In fact, our proof allows us to take for $R$ a general matrix of $SL(2,\mathbb{R})$, presented in its Cartan decomposition form, as follows.

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This research was partially conducted during the period A.A. served as a Clay Research Fellow.

**Theorem 2.** *For every $t > 0$, $s > 0$, $\alpha \in \mathbb{R}$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R = R(\theta)H(s)R(\alpha)$, if $m(w) \leq \epsilon |w| (\log|w| \log\log|w|)^{-1}$, then the spectral radius of $A_w$ is at least $e^{|w|\gamma}$.*

**Corollary.** *For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $R \in SL(2,\mathbb{R})$ with respect to the Haar measure, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R$, if $m(w) \leq \epsilon |w| (\log|w| \log\log|w|)^{-1}$, then the spectral radius of $A_w$ is at least $e^{|w|\gamma}$.*

## 2. Proof of the theorems

We now give a detailed proof of theorem 1. Then we shall indicate how theorem 2 is obtained following the same lines.

From now on we fix $t > 0$, and drop the dependence on $t$ from the notation.

For a given word $w$ we shall use the notations $w_{[i\,j]} = w_j \dots w_i$ for $1 \leq i \leq j \leq |w|$. We also let $a_w, b_w, c_w, d_w : \mathbb{R} \to \mathbb{R}$ be defined so that $A_w(\theta) = \begin{pmatrix} a_w(\theta) & b_w(\theta) \\ c_w(\theta) & d_w(\theta) \end{pmatrix}$.

Let us say that a function $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ is good if

$$\forall k, l \geq 1, \ \psi(k) + \psi(l) \leq \psi(k+l) - \log 2. \tag{1}$$

We will mostly work with multiples (by reals greater than 1) of the functions $\psi_1(m) = m(1 + \log^2 m)$ and $\psi_2(m) = m(1 + \log m)(1 + \log\log\max\{e, m\})$ (with $0 \log 0 = 0$). Both $\psi_1$ and $\psi_2$ are easily seen to be good.

Given a good function $\psi$ and $0 < \gamma \leq \frac{t}{2}$, for any word $w$ of length $n$, we let $F_w(\psi, \gamma) = F_w$ be the set of all $\theta \in [0, \pi)$ such that

$$\log|a_{w_{[1\,k]}}| \geq k\gamma - \psi(m(w_{[1\,k]})) \text{ and } \log|a_{w_{[k+1\,n]}}| \geq (n-k)\gamma - \psi(m(w_{[k+1\,n]}))$$
$$\text{for all } 0 < k < n, \quad \text{but } \log|a_w| < n\gamma - \psi(m(w)). \tag{2}$$

Notice that if $F_w$ is not empty, necessarily $w_1 = w_n = R$. In view of (1), it follows that on the set $F_w$,

$$|a_w| \leq \frac{1}{2} |a_{w_{[1\,k]}} a_{w_{[k+1\,n]}}|, \quad \forall\, 0 < k < n. \tag{3}$$

**Lemma 1.** *For every $w$ we have, writing $|w| = n$ and $m(w) = m$:*

$$|F_w| \leq 8n^2 e^{\psi([\frac{m}{2}]) + \psi(m - [\frac{m}{2}]) - \psi(m)}. \tag{4}$$

*Proof.* Since $a_\omega$ is in general a polynomial of degree $m(\omega)$ in $\cos\theta$, as is easily checked, the set $F_w$ is the union of at most $4nm$ intervals. Now, in order to bound the size of such an interval, we show that the derivative of $a_w$ with respect to $\theta$ at any (fixed) point of $F_w$ is not too small.

Since the derivative of $R(\theta)$ is $R(\frac{\pi}{2})R(\theta)$, using the product rule, it is easy to derive the following formula for the derivative of $a_w$:

$$a'_w = \sum_{k,\, w_k = R} c_{w_{[1\,k]}} a_{w_{[k+1\,n]}} - a_{w_{[1\,k]}} b_{w_{[k+1\,n]}}. \tag{5}$$

On the one hand, we have, for all $0 < k < n$,

$$a_w = a_{w_{[1\,k]}} a_{w_{[k+1\,n]}} + c_{w_{[1\,k]}} b_{w_{[k+1\,n]}}. \tag{6}$$

In view of (3), this shows that

$$\frac{1}{2} \leq -\frac{c_{w_{[1\,k]}} b_{w_{[k+1\,n]}}}{a_{w_{[1\,k]}} a_{w_{[k+1\,n]}}} \leq \frac{3}{2}. \tag{7}$$

In particular, for each $0<k<n$, $c_{w_{[1\,k]}}a_{w_{[k+1\,n]}}$ and $-a_{w_{[1\,k]}}b_{w_{[k+1\,n]}}$ have the same sign.

On the other hand, one easily sees that $\forall 1<k<n$, the upper left entry of the matrix $A_{w_{[k+1\,n]}}R(\frac{\pi}{2})A_{w_{[1\,k]}}$ is $c_{w_{[1\,k]}}a_{w_{[k+1\,n]}}-a_{w_{[1\,k]}}b_{w_{[k+1\,n]}}=c_{w_{[1\,k-1]}}a_{w_{[k\,n]}}-a_{w_{[1\,k-1]}}b_{w_{[k\,n]}}$ if $w_k=R$ and $c_{w_{[1\,k]}}a_{w_{[k+1\,n]}}-a_{w_{[1\,k]}}b_{w_{[k+1\,n]}}=e^{-t}c_{w_{[1\,k-1]}}a_{w_{[k\,n]}}-e^{t}a_{w_{[1\,k-1]}}b_{w_{[k\,n]}}$ if $w_k=H$ (indeed, $R(\frac{\pi}{2})H(t)=H(-t)R(\frac{\pi}{2})=H(t)H(-2t)R(\frac{\pi}{2})$).

After finite iteration, we deduce from these observations that the quantities $c_{w_{[1\,k]}}a_{w_{[k+1\,n]}}$ and $-a_{w_{[1\,k]}}b_{w_{[k+1\,n]}}$ for $k$ varying from 1 to $n-1$ have all the same sign; among them, the summands in (5). Therefore, taking $k$ with $w_k=R$ so that $m(w_{[1\,k]})=[\frac{m}{2}]$ where $m=m(w)$, we have

$$|a'_w|\geq |c_{w_{[1\,k]}}a_{w_{[k+1\,n]}}|+|a_{w_{[1\,k]}}b_{w_{[k+1\,n]}}|\geq 2|a_{w_{[1\,k]}}a_{w_{[k+1\,n]}}c_{w_{[1\,k]}}b_{w_{[k+1\,n]}}|^{\frac{1}{2}}.$$

From (7) and (2), we get (at any point $\theta\in F_w$):

$$\begin{aligned}|a'_w|&\geq |a_{w_{[1\,k]}}a_{w_{[k+1\,n]}}|\\&\geq e^{n\frac{t}{2}-\psi([\frac{m}{2}])-\psi(m-[\frac{m}{2}])}\end{aligned}$$

From the above minoration, we deduce that any interval in $F_w$ as defined by (2) is of length less than $2e^{\psi([\frac{m}{2}])+\psi(m-[\frac{m}{2}])-\psi(m)}$. Since $F_w$ is the union of at most $4nm$ such intervals, the result follows. □

**Lemma 2.** *If $F_w\neq\emptyset$ then*

$$n\leq m(1+\frac{1}{t}\psi(m)),$$

*where $n=|w|$ and $m=m(w)$.*

*Proof.* Let us fix some $\theta\in F_w$, and write $w=w_{[k+r+1\,n]}H^r w_{[1\,k]}$ with $r$ maximal. Since $w_1=w_n=R$, as we have already observed, one has $0<k<n-r$, $m(w_{[1\,k]}),m(w_{[k+r+1\,n]})\geq\ 1$, and

$$r\geq\frac{n-m}{m-1}. \tag{8}$$

We have

$$a_w=e^{r\frac{t}{2}}a_{w_{[1\,k]}}a_{w_{[k+r+1\,n]}}+e^{-r\frac{t}{2}}c_{w_{[1\,k]}}b_{w_{[k+r+1\,n]}}. \tag{9}$$

Observe that in general $\max(a_\omega^2+c_\omega^2,b_\omega^2+d_\omega^2)\leq e^{|\omega|t}$, so that here

$$|c_{w_{[1\,k]}}b_{w_{[k+r+1\,n]}}|\leq e^{(n-r)\frac{t}{2}}.$$

From (1),(2) and (9), we get

$$\begin{aligned}2e^{n\frac{t}{2}-\psi(m)}&\leq e^{n\frac{t}{2}-\psi(m(w_{[1\,k]}))-\psi(m(w_{[k+r+1\,n]}))}\\&\leq e^{r\frac{t}{2}}|a_{w_{[1\,k]}}a_{w_{[k+r+1\,n]}}|\\&< e^{n\frac{t}{2}-\psi(m)}+e^{(n-2r)\frac{t}{2}}.\end{aligned}$$

Hence $rt<\psi(m)$, which combined with (8) gives the result. □

From now on, let $E(\psi,\gamma)$ denote the set of all $\theta\in[0,\pi)$ such that

$$\log|a_w(\theta)|<|w|\gamma-\psi(m(w))\text{ for some word }w. \tag{10}$$

**Lemma 3.** *There exists some constant $c>0$ such that $|E(\lambda\psi_1,\frac{t}{2})|=O_{\lambda\geq 1}(e^{-c\lambda})$.*

*Proof.* Let $E_n = E_n(\lambda\psi_1, \frac{t}{2}) \subset E = E(\lambda\psi_1, \frac{t}{2})$ be the set of $\theta$ such that $n$ is the minimal length of a word $w$ such that (10) holds. Clearly $E$ is the disjoint union of the $E_n$'s and each $E_n$ is covered by the $F_w$'s with $|w| = n$.

We then apply lemmas 1 and 2 to estimate $|E_n|$ for $n \geq 2$ as follows:

$$|E_n| \leq \sum_{|w|=n} |F_w| \leq 8n^2 \sum_m \binom{n}{m} e^{\lambda(\psi_1(m-[\frac{m}{2}])+\psi_1([\frac{m}{2}])-\psi_1(m))}, \tag{11}$$

where the sum runs over the $2 \leq m \leq n$ such that $n \leq m(1 + \frac{1}{t}\lambda\psi_1(m))$, which implies $n \leq C_0\lambda m^2 \log^2 m$. Here and in the sequel, $C_0, C_1, \dots$ stand for positive constants independent of $m$, $n$ or $\lambda$.

For $n = 1$, notice that $E_1 = \{\, \theta \mid |\cos\theta| < e^{\frac{t}{2}-\lambda} \}$.

It is readily seen that $\forall m \geq 2$, $\psi_1(m - [\frac{m}{2}]) + \psi_1([\frac{m}{2}]) - \psi_1(m) \leq -C_1 m \log m$. On the other hand, by the use of Stirling's formula, we find that

$$\binom{n}{m} \leq e^{m\log n - m\log m + C_2 m}. \tag{12}$$

So, summing over $n$ in (11) and then reversing the order of summation yields

$$\begin{aligned}|E| &\leq |E_1| + \sum_{m\geq 2} e^{(C_3 - C_1\lambda)m\log m} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} \\ &\leq C_4 e^{-\lambda} + \sum_{m\geq 2} e^{(C_5 - C_1\lambda)m\log m + (m+3)\log\lambda}.\end{aligned}$$

For large $\lambda$, this sum is finite and less than $e^{-c\lambda}$. □

**Lemma 4.** *Let $0 < \gamma < \frac{t}{2}$. There exists some constant $c > 0$ such that $|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| = O_{\lambda\geq 1}(e^{-c\lambda})$.*

*Proof.* We first notice that if $F_w(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2}) \neq \emptyset$, then $\lambda\psi_1(m(w)) \geq (\frac{t}{2} - \gamma)|w|$. Thus, proceeding as in the previous lemma, we get (even for $n = 1$)

$$|E_n(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| \leq 8n^2 \sum_{\substack{\lambda\psi_1(m) \geq (\frac{t}{2}-\gamma)n \\ m\geq 2}} \binom{n}{m} e^{\lambda(\psi_2(m-[\frac{m}{2}])+\psi_2([\frac{m}{2}])-\psi_2(m))}.$$

Here $\forall m \geq 2$, $\psi_2(m - [\frac{m}{2}]) + \psi_2([\frac{m}{2}]) - \psi_2(m) \leq -C_6 m(1 + \log\log\max\{e, m\})$. Using again (12), we obtain

$$\begin{aligned}|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{t}{2})| &\leq \sum_{m\geq 2} e^{(C_7 - C_6\lambda)m(1+\log\log\max\{e,m\}) - m\log m} \sum_{n \leq C_8\lambda m \log^2 m} n^{(m+2)} \\ &\leq \sum_{m\geq 2} e^{(C_9 - C_6\lambda)m(1+\log\log\max\{e,m\}) + (m+3)\log\lambda}.\end{aligned}$$

We conclude as before. □

The lemmata 3 and 4 show that for $0 < \gamma < \frac{t}{2}$, the sum $\sum_{\lambda\in\mathbb{N}^*} |E(\lambda\psi_2, \gamma)|$ converges. By the Borel-Cantelli lemma, we conclude that for almost every $\theta$, there exists $\lambda \geq 1$ such that for all word $w$, $\log|a_w(\theta)| \geq |w|\gamma - \lambda\psi_2(m(w))$.

It follows that for almost every $\theta$, if $|w|$ is large and $m(w)$ is much smaller than $|w|(\log|w|\log\log|w|)^{-1}$, then $\frac{1}{|w|}\log\|A_w(\theta)\|$ is close to $\frac{t}{2}$, as well as $\frac{1}{|w|^2}\log\|A_{ww}(\theta)\|$. But

$$A_{ww}(\theta) - A_w \mathrm{tr} A_w + \mathrm{id} = A_w(\theta)^2 - A_w \mathrm{tr} A_w + \mathrm{id} = 0,$$

since $A_w \in \mathrm{SL}(2,\mathbb{R})$, which shows that $\frac{1}{|w|}\log|\mathrm{tr}A_w|$ is close to $\frac{t}{2}$, yielding the estimate on the spectral radius in theorem 1.

In order to prove theorem 2 by the same method, we consider, instead of the words on $H$ and $R$, words $w = w_n \dots w_1$ on $H(t)$, $R(\theta)$, $H(s)$ and $R(\alpha)$ such that the last three ones always appear consecutively, except maybe at the ends of the word, and $m(w)$ is now the number of these occuring in $w$. Then the proof goes the same way, notably the considerations of sign in lemma 1.

## References

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